GENERALIZED-LUSH SPACES AND THE MAZUR-ULAM PROPERTY

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Abstract. We introduce a new class of Banach spaces, called generalized-lush spaces (GL-spaces for short), which contains almost-CL-spaces, separable lush spaces (specially, separable C-rich subspaces of $C(K)$), and even the two-dimensional space with hexagonal norm. We obtain that the space $C(K,E)$ of the vector-valued continuous functions is a GL-space whenever E is, and show that the GL-space is stable under c_0-, l_1- and l_∞-sums. As an application, we prove that the Mazur-Ulam property holds for a larger class of Banach spaces, called local-GL-spaces, including all lush spaces and GL-spaces. Furthermore, we generalize the stability properties of GL-spaces to local-GL-spaces. From this, we can obtain many examples of Banach spaces having the Mazur-Ulam property.

1. Introduction

The classical Mazur-Ulam theorem states that every surjective isometry between normed spaces is a linear mapping up to translation. In 1972, Mankiewiz [18] extended this by showing that every surjective isometry between the open connected subsets of normed spaces can be extended to a surjective affine isometry on the whole space. This result implies that the metric structure on the unit ball of a real normed space constrains the linear structure of the whole space. It is of interest to us whether this result can be extended to unit spheres. In 1987, Tingley [29] first studied isometries on the unit sphere and raised the isometric extension problem:

Problem 1.1. Let E and F be normed spaces with the unit spheres S_E and S_F, respectively. If $T : S_E \to S_F$ is a surjective isometry, then does there exist a linear isometry $\widetilde{T} : E \to F$ such that $\widetilde{T}|_{S_E} = T$?

There is a number of publications on this topic and many positive answers on special spaces, for example, $l^p(\Gamma)$, $L^p(\mu)(0 < p \leq \infty)$, $C(K)$, even the James spaces and the (modified) Tsirelson spaces (see [4, 5, 16, 17, 24–27] and the references therein).

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Recently in [3], Cheng and Dong considered the extension question of isometries between unit spheres of Banach space and introduced the Mazur-Ulam property:

Definition 1.2. A Banach space E is said to have the *Mazur-Ulam property* (briefly MUP) provided that for every Banach space F, every surjective isometry T between the two unit spheres of E and F is the restriction of a linear isometry between the two spaces.

Cheng and Dong attacked the problem for the class of CL-spaces admitting a smooth point and polyhedral spaces. Unfortunately their interesting attempt failed by a mistake at the very end of the proof (also see the introduction in [10, 28]). In [10], Kadets and Martín proved that finite-dimensional polyhedral Banach spaces have the MUP. Notice that the problem is still open even in two dimension case. In [28], the authors Tan and Liu proved that every almost-CL-spaces admitting a smooth point (specially, separable almost-CL-spaces) has the MUP.

Recall that R. Fullerton [8] first introduced the notion of CL-spaces. It was extended by Lima [14, 15] who introduced the almost-CL-spaces and gave the examples of real CL-spaces which are $L_1(\mu)$ and its isometric preduals, in particular $C(K)$, where $C(K)$ is a compact Hausdorff space. The infinite-dimensional complex $L_1(\mu)$ spaces are proved by Martín and Payá [20] to be only almost-CL-spaces. Lush spaces were introduced recently in [1] and have been extensively studied recently in [2, 10, 11]. Such spaces are of importance to supply an example of a Banach space E with the numerical index $n(E) < n(E^*)$. It thus gives a negative answer to a question which has been latent since the beginning of the theory of numerical indices in the seventies. Now, a natural and interesting question is: "*Does every almost-CL-space, even every lush space, has the MUP?*"

In this paper, we introduce a natural concept of generalized-lush spaces (GL-spaces for short), which contains almost-CL-spaces, separable lush spaces (specially, separable C-rich subspaces of $C(K)$), and even the two-dimensional space with hexagonal norm. We obtain that the space $C(K, E)$ of the vector-valued continuous functions is a generalized-lush space whenever E is, and show the stability of generalized-lush spaces by c_0-, l_1- and l_∞-sums. Then we prove that the Mazur-Ulam property holds for a larger class of Banach spaces than GL-spaces, called local-GL-spaces, including all lush spaces and GL-spaces. Furthermore, we show that the $C(K, E)$ is a local-GL-space whenever E is, and the stability by c_0-, l_1- and l_∞-sums also holds for local-GL-spaces.

Throughout this paper, we consider the spaces all over the real field. For a Banach space E, B_E, S_E, E^* and $L(E)$ will stand for the unit ball of E, the unit sphere of E, the dual space and the Banach algebra of all bounded linear operators on E. A slice is a subset of B_E of the form

$$S(x^*, \alpha) = \{x \in B_E : Re\, x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{E^*}$ and $0 < \alpha < 1$.

We recall here some basic concepts.

Definition 1.3. Let E be a Banach space.

(1) E is said to be a *CL-space* if for every maximal convex set C of S_E, we have $B_E = co(C \cup -C)$.
(2) E is said to be an *almost-CL-space* if for every maximal convex set C of S_E, we have $B_E = \overline{co}(C \cup -C)$.
(3) E is said to be *lush* if for every $x, y \in S_E$ and every $\varepsilon > 0$, there exists a slice $S = S(x^*, \varepsilon)$ such that $x \in S$ and $\operatorname{dist}(y, \operatorname{aco}(S)) < \varepsilon$.

It is an evident implication that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and none of the one-way implications can be reversed (see [20, Proposition 1] and [1, Example 3.4]).

The numerical index of a Banach space E was first suggested by G. Lumer in 1968 (see [6]), and it is the constant $n(E)$ defined by

$$\begin{aligned} n(E) &= \inf\{v(T) : T \in L(E), \|T\| = 1\} \\ &= \max\{k \geq 0 : k\|T\| \leq v(T) \;\; T \in L(E)\}, \end{aligned}$$

where $v(T)$ is the numerical radius of T and is given by

$$v(T) = \sup\{|x^*(T(x))| : x \in S_E, x^* \in S_{E^*}, x^*(x) = 1\}.$$

More information and background on numerical indices can be found in the recent survey [12] and references therein.

2. Generalized-lush spaces

The aim of this section is to study generalized-lush spaces (GL-spaces for short). We present many examples and prove a stronger property for separable GL-spaces, and we also show that GL-spaces have some stability properties.

Definition 2.1. A Banach space E is said to be a *generalized-lush space* (GL-space) if for every $x \in S_E$ and every $\varepsilon > 0$ there exists a slice $S = S(x^*, \varepsilon)$ with $x^* \in S_{E^*}$ such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon$$

for all $y \in S_E$.

The following proposition for separable GL-spaces is based on an idea from [10, Lemma 4.2], and it is of independent interest.

Proposition 2.2. *Let E be a separable GL-space and $G \subset S(E^*)$ a norming subset for E. Then for every $\varepsilon > 0$ the set*

$$\{x^* \in G : dist(y, S) + dist(y, -S) < 2 + \varepsilon \;\; \textit{for all } y \in S_E, \;\; \textit{where } S = S(x^*, \varepsilon)\}$$

is a weak* G_δ-dense subset of the weak* closure of G.

Proof. Let $(y_n) \subset S_E$ be a sequence dense in S_E. Fix $0 < \varepsilon < 1$. Given $n \geq 1$, set

$$K_n = \{x^* \in G : \quad \mathrm{dist}(y_n, S) + \mathrm{dist}(y_n, -S) < 2 + \varepsilon \quad \text{where} \quad S = S(x^*, \varepsilon)\}.$$

Then K_n is weak*-open and $\overline{K_n}^{\omega^*} = \overline{G}^{\omega^*}$. Indeed, if $x^* \in K_n$, then there exist $x_n \in S(x^*, \varepsilon)$ and $z_n \in -S(x^*, \varepsilon)$ such that

$$\|x_n - y_n\| + \|y_n - z_n\| < 2 + \varepsilon.$$

Let

$$U = \{y^* \in G : y^*(x_n) > 1 - \varepsilon \quad \text{and} \quad y^*(-z_n) > 1 - \varepsilon\}.$$

Then it is easily checked that U is a weak*-neighborhood of x^* in G satisfying $U \subset K_n$. Thus K_n is weak*-open.

To prove $\overline{K_n}^{\omega^*} = \overline{G}^{\omega^*}$, it is enough to show that $G \subset \overline{K_n}^{\omega^*}$. Since [7, Lemma 3.40] states that for every $x^* \in G$, the weak*-slices containing x^* form a neighborhood base of x^*, it suffices to prove that the weak*-slice $S(x, \varepsilon_1) \cap K_n \neq \emptyset$ for all $\varepsilon_1 \in (0, \varepsilon)$. Since E is a GL-space, there is a slice $S = S(y^*, \varepsilon_1/3)$ such that

$$x \in S \quad \text{and} \quad \mathrm{dist}(y_n, S) + \mathrm{dist}(y_n, -S) < 2 + \varepsilon_1.$$

Thus we may find $x'_n \in S$ and $z'_n \in -S$ such that

$$\|x'_n - y_n\| + \|y_n - z'_n\| < 2 + \varepsilon_1 \quad \text{and} \quad \|x + x'_n - z'_n\| > 3 - \varepsilon_1.$$

Note that G is a norming subset of S_{E^*}. Thus there is a $z^* \in G$ such that

$$z^*(x + x'_n - z'_n) = \|x + x'_n - z'_n\| > 3 - \varepsilon_1.$$

This implies that $z^* \in S(x, \varepsilon_1) \cap K_n$.

Now set $K = \bigcap_{n \in \mathbb{N}} K_n$. Then by the Baire theorem, K is a weak* G_δ-dense subset of $\overline{G}^{\omega^*}$. This together with density of (y_n) in S_E gives the desired conclusion. $\square$

As a consequence, we have a stronger characterization for separable GL-spaces which indicates that the x^* in the definition of GL-spaces can be chosen from $\mathrm{ext}(B_{E^*})$.

Corollary 2.3. *Let E be a separable Banach space. Then E is a GL-space if and only if for every $x \in S_E$ and every $\varepsilon > 0$ there exists a slice $S = S(x^*, \varepsilon)$ with $x^* \in$ ext(B_{E^*}) such that*

$$x \in S \quad \text{and} \quad dist(y, S) + dist(y, -S) < 2 + \varepsilon$$

for all $y \in B_E$.

Now we have the following important examples.

Example 2.4. Every almost-CL-space is a GL-space.

Proof. Let E be an almost-CL-space. For every $x \in S_E$ and $\varepsilon > 0$, there exists a maximal convex set C of S_E such that $x \in C$. Choose $f \in S_{E^*}$ such that $f(z) = 1$ for every $z \in C$, and set $S = S(f, \varepsilon)$. Then $C \subset S$. Since E is an almost-CL-space, it follows that $B_E = \overline{co}(S \cup -S)$. So for every $y \in S_E$, there are $\lambda \in [0, 1]$, $y_1 \in S$ and $y_2 \in -S$ such that

$$\|\lambda y_1 + (1 - \lambda)y_2 - y\| < \varepsilon/2.$$

This leads to

$$\|y_1 - y\| + \|y_2 - y\| < 2 + \varepsilon,$$

which completes the proof. □

Since all $C(K)$, real $L_1(\mu)$ are CL-spaces (in particular, almost-CL-spaces), they are GL-spaces. Moreover, according to [10, Theorem 4.3] showing that the separable lush space enjoys a stronger property, we can have a larger class of spaces which are GL-spaces, and they are not almost-CL-spaces in general (see, [1, Example 3.4]).

Example 2.5. Every separable lush space is a GL-space.

Proof. Note that [10, Theorem 4.3] implies that if E is a separable lush space, then there is a norming subset K of S_{E^*} such that

$$B_E = \overline{co}(S(x^*, \varepsilon) \cup -S(x^*, \varepsilon))$$

for every $x^* \in K$ and every $\varepsilon > 0$. A similar analysis as in Example 2.4 yields the desired conclusion. □

Let K be a compact Hausdorff space. A closed subspace X of $C(K)$ is said to be *C-rich* if for every nonempty open subset U of K and every $\varepsilon > 0$, there is a positive function h with $\|h\| = 1$ and $\text{supp}(h) \subset U$ such that $\text{dist}(h, X) < \varepsilon$. This definition covers all finite-codimensional subspaces of $C[0, 1]$ (see [1, Proposition 2.5]) and a subspace X of $C[0, 1]$ whenever $C[0, 1]/X$ does not contain a copy of $C[0, 1]$ (see [13, Proposition 1.2 and Definition 2.1]). For more examples and results about C-rich subspaces we refer to [2, 10, 11] and references therein. Notice that all C-rich subspaces of $C(K)$ have been proved in [1, Theorem 2.4] to be lush. Therefore we get the following example.

Example 2.6. Every C-rich separable subspace of $C(K)$ is a GL-space.

Observe that all the above examples of GL-spaces are Banach spaces with numerical index 1. We remark from the following examples that there may exist many GL-spaces whose numerical index are not 1. The two-dimensional space with hexagonal norm is firstly introduced by M. Martín and J. Meri [19].

Example 2.7. The space $E = (\mathbb{R}^2, \|\cdot\|)$ whose norm is given by

$$\|(\xi, \eta)\| = \max\{|\eta|, |\xi| + 1/2|\eta|\} \quad \forall (\xi, \eta) \in E,$$

with numerical index 1/2 is a GL-space.

Proof. It is shown by [19, Theorem 1] that E has numerical index $1/2$. To prove that E is a GL-space, given $x = (a, b) \in S_E$ and $\varepsilon > 0$, we divide the proof into two cases. By symmetry considerations, we assume that $a, b \geq 0$.

Case 1: $b = 1$. Define a functional $f \in S_{E^*}$ by $f(z) = \eta$ for all $z = (\xi, \eta) \in E$. Set $S = S(f, \varepsilon)$. Then $x \in S$, and for every $y = (c, d) \in S_E$, consider the two vectors

$$y_1 = (c, 1) \text{ and } y_2 = (c, -1).$$

We clearly have $y_1 \in S$ and $y_2 \in -S$, and moreover,

$$\|y - y_1\| + \|y - y_2\| = 2 < 2 + \varepsilon.$$

Case 2: $b < 1$. We make the convention that $\operatorname{sign}(0) = 1$. Let $f \in S_{E^*}$ be defined by $f(z) = \xi + \eta/2$ for every $z = (\xi, \eta) \in E$. This guarantees that $x \in S = S(f, \varepsilon)$. For every $y = (c, d) \in S_E$, we set

$$\begin{cases} y_1 = (\operatorname{sign}(c), 0),\ y_2 = \operatorname{sign}(d)(1/2, 1) & \text{if } cd \leq 0; \\ y_1 = -(\operatorname{sign}(c), 0),\ y_2 = \operatorname{sign}(d)(1/2, 1) & \text{if } cd > 0 \text{ and } |d| = 1; \\ y_1 = y,\ y_2 = -y, & \text{if } cd > 0 \text{ and } |d| < 1. \end{cases}$$

Then $y_1, y_2 \in S \cup (-S)$ satisfy

$$\|y - y_1\| + \|y - y_2\| = 2 < 2 + \varepsilon.$$

We thus complete the proof. □

By Example 2.7, the following Theorems 2.10, 2.11 and [21, Proposition 1] which shows that the numerical index of the c_0-,l_1-,or l_∞-sum of Banach spaces is the infimum numerical index of the summands, we may construct more examples of specific GL-spaces with numerical index $1/2$.

Example 2.8. The space $E = (c_0, \|\cdot\|)$ equipped with the norm

$$\|x\| = \max\{\sup_{k \in \mathbb{N}} |\xi_k|, |\xi_1| + 1/2|\xi_2|\} \quad \forall x = (\xi_k) \in E$$

is a GL-space with numerical index $1/2$.

Proof. It is actually the space $c_0 \bigoplus_\infty X$ where X is just the hexagonal space in Example 2.7. □

We shall give an observation that in the definition of GL-spaces we can take y to be in the unit ball instead of being in the unit sphere. With the help of this observation, one can check whether the space being considered is a GL-space in an easier way. We will use it later to get some stability properties of GL-spaces.

Lemma 2.9. *If E is a GL-space, then for every $x \in S_E$ and every $\varepsilon > 0$ there exists a slice $S = S(x^*, \varepsilon)$ with $x^* \in S_{E^*}$ such that*

$$x \in S \quad \textit{and} \quad \mathit{dist}(y, S) + \mathit{dist}(y, -S) < 2 + \varepsilon$$

for all $y \in B_E$.

Proof. For every $x \in S_E$ and every $\varepsilon > 0$, let $S = S(x^*, \varepsilon)$ be such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(z, S) + \operatorname{dist}(z, -S) < 2 + \varepsilon$$

for all $z \in S_E$. Given $y \in B_E$, since the case where $y = 0$ is trivial, we may assume that $y \neq 0$. Then there exist $u, -v \in S$ such that

$$\|u - \frac{y}{\|y\|}\| + \|v - \frac{y}{\|y\|}\| < 2 + \varepsilon.$$

Triangle inequality hence yields

$$\|u - y\| + \|v - y\| < 2 + \varepsilon\|y\| \leq 2 + \varepsilon.$$

This completes the proof. $\square$

Given a compact Hausdorff space K and a Banach space E, we denote by $C(K, E)$ the Banach space of all continuous functions from K into E, endowed with its natural supremum norm.

Theorem 2.10. *Let K be a compact Hausdorff space and E a GL-space, then $C(K, E)$ is a GL-space.*

Proof. Given $f \in S_{C(K,E)}$ and $\varepsilon > 0$, there exists a $t_0 \in K$ such that $\|f(t_0)\| = 1$. Since E is a GL-space, it follows from this and Lemma 2.9 that there is an $x^* \in S_E^*$ with $S_{x^*} = S(x^*, \varepsilon/2)$ such that

$$f(t_0) \in S_{x^*} \ \text{ and } \ \operatorname{dist}(y, S_{x^*}) + \operatorname{dist}(y, -S_{x^*}) < 2 + \varepsilon/2$$

for all $y \in B_E$. Define a functional $f^* \in S_{C(K,E)^*}$ by $f^*(g) = x^*(g(t_0))$ for every $g \in C(K, E)$, and put $S = S(f^*, \varepsilon)$. For every $g \in S_{C(K,E)}$, we have $g(t_0) \in B_E$. Thus there are $y_1 \in S_{x^*}$ and $y_2 \in -S_{x^*}$ such that

$$\|g(t_0) - y_1\| + \|g(t_0) - y_2\| < 2 + \varepsilon/2.$$

Then we can build a continuous map $\phi : K \to [0, 1]$ defined by

$$\phi(t_0) = 1 \ \text{ and } \ \phi(t) = 0 \ \text{ if } \ \|g(t) - g(t_0)\| \geq \varepsilon/4.$$

Consider $h_1 \in S$ and $h_2 \in -S$ given by

$$h_i(t) = \phi(t)y_i + (1 - \phi(t))g(t)(i = 1, 2) \ \text{ for every } t \in K.$$

Then it is trivial to see that

$$\|g - h_1\| + \|g - h_2\| < 2 + \varepsilon.$$

Hence $C(K, E)$ is a GL-space. $\square$

For more examples of GL-spaces, we need discuss the stability of GL-spaces by c_0-, l_1- and l_∞-sums. Recall that the c_0-sum (resp. l_1-sum and l_∞-sum) of a family of Banach spaces $\{E_\lambda : \lambda \in \Lambda\}$ are denoted by $[\bigoplus_{\lambda \in \Lambda} E_\lambda]_{c_0}$ (resp. $[\bigoplus_{\lambda \in \Lambda} E_\lambda]_{l_1}$ and $[\bigoplus_{\lambda \in \Lambda} E_\lambda]_{l_\infty}$).

Theorem 2.11. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, and let $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_F$ where $F = c_0$ or l_1. Then E is a GL-space if and only if each E_λ is a GL-space.*

Proof. Note that $E^* = [\bigoplus_{\lambda\in\Lambda} E_\lambda^*]_{l_1}$ if $F = c_0$ and $E^* = [\bigoplus_{\lambda\in\Lambda} E_\lambda^*]_{l_\infty}$ if $F = l_1$. This fact will be used without comment in the following proof.

In the c_0-sum case, we first show the "if" part. Fix $x = (x_\lambda) \in S_E$ and $\varepsilon > 0$. We may find a λ_0 such that $\|x_{\lambda_0}\| = 1$. Since E_{λ_0} is a GL-space, by Lemma 2.9 there is a slice $S_{\lambda_0} = S(x^*_{\lambda_0}, \varepsilon) \subset B_{E_{\lambda_0}}$ with $x^*_{\lambda_0} \in S_{E^*_{\lambda_0}}$ such that

$$x_{\lambda_0} \in S_{\lambda_0} \ \text{ and } \ \operatorname{dist}(z, S_{\lambda_0}) + \operatorname{dist}(z, -S_{\lambda_0}) < 2 + \varepsilon$$

for all $z \in B_{E_{\lambda_0}}$. Choose $x^* = (x^*_\lambda) \in S_{E^*}$ with $x^*_\lambda = 0$ for all $\lambda \neq \lambda_0$, and let $S = S(x^*, \varepsilon)$. Then $x \in S$, and it is easy to see from the definition of E that

$$\operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon \tag{2.1}$$

for all $y \in S_E$. Thus E is a GL-space.

Now we deal with the "only if" part. For every $\lambda \in \Lambda$, fix $x_\lambda \in S_{E_\lambda}$ and $\varepsilon > 0$. Take $x = (x_\delta) \in S_E$ with $x_\delta = 0$ for all $\delta \neq \lambda$. Then $x \in S_E$, and thus there exists an $x^* = (x^*_\delta) \in S_{E^*}$ with $S = S(x^*, \varepsilon/2)$ such that

$$x \in S \ \text{ and } \ \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon/2 \tag{2.2}$$

for all $y \in S_E$. Note that $x_\lambda \in S_\lambda = S(x^*_\lambda/\|x^*_\lambda\|, \varepsilon)$. To show that E_λ is a GL-space, it remains to check that for all $y_\lambda \in S_{E_\lambda}$

$$\operatorname{dist}(y_\lambda, S_\lambda) + \operatorname{dist}(y_\lambda, -S_\lambda) < 2 + \varepsilon.$$

Now given $y_\lambda \in S_{E_\lambda}$, consider $y = (y_\delta) \in S_E$ with $y_\delta = 0$ for all $\delta \neq \lambda$. By (2.2), there are $u = (u_\delta) \in S$ and $v = (v_\delta) \in -S$ such that

$$\|y - u\| + \|y - v\| < 2 + \varepsilon/2.$$

The definition of E thus gives

$$\|y_\lambda - u_\lambda\| + \|y_\lambda - v_\lambda\| < 2 + \varepsilon/2.$$

Observe that $\|x^*_\lambda\| \geq x^*_\lambda(x_\lambda) > 1 - \varepsilon/2$, and therefore $\sum_{\delta\neq\lambda} \|x^*_\delta\| < \varepsilon/2$. So

$$x^*_\lambda(u_\lambda) > 1 - \varepsilon/2 - \sum_{\delta\neq\lambda} \|x^*_\delta\| > 1 - \varepsilon.$$

Similarly, $x^*_\lambda(-v_\lambda) > 1 - \varepsilon$. Hence E_λ is a GL-space.

In the l_1-sum case, let us prove the "if" part. Given $x = (x_\lambda) \in S_E$ and $\varepsilon > 0$, for each λ with $x_\lambda \neq 0$, there is a corresponding slice $S_\lambda = S(x^*_\lambda, \varepsilon)$ with $x^*_\lambda \in S_{E^*_\lambda}$ such that

$$x^*_\lambda(x_\lambda) > (1 - \varepsilon)\|x_\lambda\| \ \text{and} \ \operatorname{dist}(z_\lambda, S_\lambda) + \operatorname{dist}(z_\lambda, -S_\lambda) < 2 + \varepsilon$$

for all $z_\lambda \in S_{E_\lambda}$. Then $x^* = (x^*_\lambda) \in S_{E^*}$ with $x^*_\lambda = 0$ whenever $x_\lambda = 0$, and the required slice satisfying (2.1) is $S(x^*, \varepsilon)$. Therefore E is a GL-space.

For the "only if" part, fix $x_\lambda \in S_{E_\lambda}$ and $0 < \varepsilon < 1/2$. Then $x = (x_\delta) \in S_E$ where $x_\delta = 0$ for all $\delta \neq \lambda$. Since E is a GL-space, there is an $x^* = (x^*_\delta) \in S_{E^*}$ with $S = S(x^*, \varepsilon/4)$ such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon/4$$

for all $y \in S_E$. We shall prove that the slice $S_\lambda = S(x^*_\lambda/\|x^*_\lambda\|, \varepsilon)$ is the desired one, namely that $x_\lambda \in S_\lambda$ and $\operatorname{dist}(y_\lambda, S_\lambda) + \operatorname{dist}(y_\lambda, -S_\lambda) < 2 + \varepsilon$ for all $y_\lambda \in S_{E_\lambda}$.

It is easily checked that $x_\lambda \in S_\lambda$. For every $y_\lambda \in S_{E_\lambda}$, since $y = (y_\delta)$ is in S_E where $y_\delta = 0$ for all $\delta \neq \lambda$, there are $u = (u_\delta) \in S$ and $v = (v_\delta) \in -S$ such that

$$\|y - u\| + \|y - v\| < 2 + \varepsilon/4. \tag{2.3}$$

It follows from the definition of E that

$$\begin{aligned} \|y - u\| + \|y - v\| &= \|y_\lambda - u_\lambda\| + \sum_{\delta \neq \lambda} \|u_\delta\| + \|y_\lambda - v_\lambda\| + \sum_{\delta \neq \lambda} \|v_\delta\| \\ &> \|y_\lambda - u_\lambda\| + 1 - \varepsilon/4 - \|u_\lambda\| + \|y_\lambda - v_\lambda\| + 1 - \varepsilon/4 - \|v_\lambda\| \\ &= \|y_\lambda - u_\lambda\| - \|u_\lambda\| + \|y_\lambda - v_\lambda\| - \|v_\lambda\| + 2 - \varepsilon/2. \end{aligned} \tag{2.4}$$

We deduce from (2.3) and (2.4) that

$$\|u_\lambda\| > 1/2 - \varepsilon/2 \quad \text{and} \quad \|v_\lambda\| > 1/2 - \varepsilon/2.$$

Hence

$$x^*_\lambda(u_\lambda) > 1 - \varepsilon/4 - \sum_{\delta \neq \lambda} \|u_\delta\| \geq 1 - \varepsilon/4 - 1 + \|u_\lambda\| \geq (1 - \varepsilon)\|u_\lambda\|,$$

and similarly,

$$x^*_\lambda(-v_\lambda) > (1 - \varepsilon)\|v_\lambda\|.$$

So $w_\lambda = u_\lambda/\|u_\lambda\|$ and $t_\lambda = -v_\lambda/\|v_\lambda\|$ are in S_λ. The desired estimate

$$\|y_\lambda - w_\lambda\| + \|y_\lambda + t_\lambda\| < 2 + \varepsilon$$

which follows from (2.4) completes the proof. □

We also have a proposition establishing that the class of GL-spaces is stable under the l_∞-sum, and we omit the proof since it is just a slight modification of the "if" part in the c_0-case.

Proposition 2.12. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of GL-spaces, and let $E = [\bigoplus_{\lambda \in \Lambda} E_\lambda]_{l_\infty}$. Then E is a GL-space.*

3. The Mazur-Ulam property for local-GL-spaces

The main aim of this section is to prove that a larger class of Banach spaces have the Mazur-Ulam property. We begin with a proposition which is the key step to prove Theorem 3.7.

Proposition 3.1. *Let E, F be Banach spaces, and let $T : S_E \to S_F$ be an isometry (not necessarily surjective). If E is a GL-space, then we have*

$$\|T(x) - \lambda T(y)\| \geq \|x - \lambda y\| \quad \text{for all } x, y \in S_E \text{ and } \lambda \geq 0.$$

Proof. Given $x, y \in S_E$ with $x \neq y$ and $\lambda > 0$, set

$$z = \frac{x - \lambda y}{\|x - \lambda y\|}.$$

Since E is a GL-space, given $\varepsilon > 0$, there exists a functional $f \in S_{E^*}$ with $S = S(f, \varepsilon)$ such that

$$z \in S \quad \text{and} \quad \operatorname{dist}(w, S) + \operatorname{dist}(w, -S) < 2 + \varepsilon$$

for all $w \in S_E$. Therefore, there exist $x_1, y_1 \in S$ and $x_2, y_2 \in -S$ such that

$$\|x - x_1\| + \|x - x_2\| < 2 + \varepsilon \quad \text{and} \quad \|y - y_1\| + \|y - y_2\| < 2 + \varepsilon.$$

Then

$$2 - 2\varepsilon < f(x_1) - f(x) + f(x) - f(x_2) \leq \|x - x_1\| + \|x - x_2\| < 2 + \varepsilon.$$

This implies that

$$f(x_1) - f(x) \geq \|x - x_1\| - 3\varepsilon. \tag{3.1}$$

A similar analysis gives

$$f(y) - f(y_2) \geq \|y - y_2\| - 3\varepsilon. \tag{3.2}$$

For $i = 1$ or 2, replace x_i by $x_i/\|x_i\|$ and y_i by $y_i/\|y_i\|$ respectively if necessary we may assume that x_i and y_i have norm 1. Then there exists a functional $g \in S_{F^*}$ such that

$$g(T(x_1)) - g(T(y_2)) = \|T(x_1) - T(y_2)\| = \|x_1 - y_2\| > 2 - 2\varepsilon.$$

It follows that

$$g(T(x_1)) > 1 - 2\varepsilon \quad \text{and} \quad g(T(y_2)) < -1 + 2\varepsilon.$$

Thus by (3.1) and (3.2), we have

$$\begin{aligned} f(x) &\leq f(x_1) - \|x - x_1\| + 3\varepsilon \\ &\leq 1 - \|T(x) - T(x_1)\| + 3\varepsilon \\ &\leq 1 - (g(T(x_1)) - g(T(x)) + 3\varepsilon \\ &\leq g(T(x)) + 5\varepsilon \end{aligned}$$

and

$$\begin{aligned} f(y) &\geq f(y_2) + \|y - y_2\| - 3\varepsilon \\ &\geq -1 + \|T(y) - T(y_2)\| - 3\varepsilon \\ &\geq -1 + (g(T(y)) - g(T(y_2)) - 3\varepsilon \\ &\geq g(T(y)) - 5\varepsilon. \end{aligned}$$

As a consequence,

$$\begin{aligned} \|x - \lambda y\|(1-\varepsilon) < f(x - \lambda y) &\leq g(T(x)) + 5\varepsilon - \lambda g(T(y)) + 5\lambda\varepsilon \\ &\leq \|T(x) - \lambda T(y)\| + (5 + 5\lambda)\varepsilon. \end{aligned}$$

Since ε can be arbitrarily small, we complete the proof. □

Theorem 3.2. *Every GL-space E has the MUP.*

Proof. Let F be a Banach space, and let $T : S_E \to S_F$ be a surjective isometry. We need to show that T can be extended to a linear surjective isometry from E onto F. We first claim that for all $x, y \in S_E$ and $\lambda \geq 0$.

$$\|T(x) - \lambda T(y)\| = \|x - \lambda y\|. \tag{3.3}$$

Otherwise by Proposition 3.1, there exist $\lambda_0 > 0, x_0, y_0 \in S_E$ such that

$$\|T(x_0) - \lambda_0 T(y_0)\| > \|x_0 - \lambda_0 y_0\|. \tag{3.4}$$

Replace λ_0 by $1/\lambda_0$ if necessary we may assume that $\lambda_0 < 1$. Since $\|\lambda_0 T(y_0)\| = \lambda_0 < 1$, there exists $T(v) \in S_F$ with $v \in S_E$ such that $\lambda_0 T(y_0)$ belongs to the segment $(T(x_0), T(v))$ of B_F. By (3.4) and Proposition 3.1 we have

$$\begin{aligned} \|v - x_0\| = \|T(v) - T(x_0)\| &= \|T(v) - \lambda_0 T(y_0)\| + \|\lambda_0 T(y_0) - T(x_0)\| \\ &> \|v - \lambda_0 y_0\| + \|\lambda_0 y_0 - x_0\| \\ &\geq \|v - x_0\|. \end{aligned}$$

It is a contradiction. Now we may define the required extension $\widetilde{T}$ of T by

$$\widetilde{T}(x) = \begin{cases} \|x\| T(\frac{x}{\|x\|}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

It is easily seen from (3.3) that $\widetilde{T} : E \to F$ is a surjective isometry whose restriction to the unit sphere S_E is just T. The Mazur-Ulam theorem hence shows that $\widetilde{T}$ is linear as desired. The proof is complete. □

Note that the technique in the proof of Theorem 3.2 is still valid in more general case. We now state a result here since it will be of use later.

Proposition 3.3. *Let E, F be Banach spaces, and let $T : S_E \to S_F$ be a surjective isometry such that*

$$\|T(x) - \lambda T(y)\| \geq \|x - \lambda y\| \quad \textit{for all } x, y \in S_E \textit{ and } \lambda \geq 0.$$

Then E has the MUP.

Now we introduce a class of spaces called local-GL-spaces (including GL-spaces and lush spaces) which have the MUP. This definition is a weakening of the notion of lush spaces in the real case. We can see from the above Example 2.7 that this weakening is strict.

Definition 3.4. A Banach space E is said to be a *local-GL-space* if for every separable subspace $X \subset E$, there is a GL-subspace $Y \subset E$ such that $X \subset Y \subset E$.

Example 3.5. GL-spaces are local-GL-spaces.

The equivalent definition of lush space [2, Theorem 4.2] proves the following.

Example 3.6. Lush spaces are local-GL-spaces.

We now present the main result of this section.

Theorem 3.7. *Every local-GL-space has the MUP.*

Proof. Let E be a local-GL-space, F a Banach space and $T : S_E \to S_F$ a surjective isometry. We next show that T can be extended to a linear surjective isometry from E onto F.

Fix $x, y \in S_E$. Let $X = \mathrm{span}(x, y)$. Since E is a local-GL-space, there is a GL-space $Y \subset E$ such that $X \subset Y$. We consider T to be an isometry from S_Y to S_F. Then Propositions 3.1 and 3.3 clearly lead to the fact that T can be extended to a linear surjective isometry from E onto F. $\square$

We emphasize two evident consequences of the above theorem.

Corollary 3.8. *Every lush space has the MUP.*

Corollary 3.9. *Every C-rich subspace of $C(K)$ has the MUP.*

By the following properties, we can get more examples of spaces having the MUP.

Proposition 3.10. *If E is a local-GL-space, then $C(K, E)$ is a local-GL-space.*

Proof. Let X be a separable subspace of $C(K, E)$. We shall prove that the set

$$E_X = \bigcup_{t \in K} \{f(t) : f \in X\}$$

is a separable subset of E. Let $\{f_n\}$ be a dense sequence of X. Given $n, m \geq 1$ and $s \in K$, set $V_{s,m,n} = \{t \in K : \|f_n(t) - f_n(s)\| < 1/m\}$. The compactness

of K implies that there is a finite subset $\{s_i^{m,n} : i = 1, \cdots, k_{m,n}\}$ of K such that $K = \bigcup_{i=1}^{k_{m,n}} V_{s_i^{m,n}, m, n}$. Then it is an elementary check that the set

$$M = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{f_n(s_i^{m,n}) : i = 1, \cdots k_{m,n}\}$$

is a dense subset of E_X. It follows that $N_X = \overline{\text{span}\{E_X\}}$ is a separable subspace of E. Note that the E is a local-GL-space. So we may find a GL-space M_X such that $N_X \subset M_X \subset E$.

Let $Y = C(K, M_X)$. Then $X \subset Y$, and Theorem 2.10 shows that Y is a GL-space. This completes the proof. □

Corollary 3.11. *Let E be a local-GL-space and K be a compact Hausdorff space. Then $C(K, E)$ has the MUP.*

The proof of Theorem 2.11 can be adapted to yield a characterization of the c_0-, l_1-sums of lush spaces in both real and complex cases. The "if" part of it has been noted in [2, Propsosition 5.3], and the "only if" part is probably known but we include an argument here (as we do not find it explicitly stated in the literature).

Proposition 3.12. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, and let $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_F$ where $F = c_0$ or l_1. Then E is a lush space if and only if E_λ is a lush space for every $\lambda \in \Lambda$.*

Proof. It has been proved in [2, Propsosition 5.3] that each E_λ is a lush space, then E is also lush. We only check the "only if" statement. Note that the c_0-case follows from the proof of Theorem 2.11 with minor modifications. We omit the proof, leaving routine details to the readers.

Now for the l_1-case, fix $x_\lambda, y_\lambda \in S_{E_\lambda}$ and $0 < \varepsilon < 1/2$. Consider $x = (x_\delta), y = (y_\delta) \in S_E$ with $x_\delta = y_\delta = 0$ for all $\delta \neq \lambda$. Then there is an $x^* = (x_\delta^*) \in S_{E^*}$ with $S = S(x^*, \varepsilon/8)$ such that

$$x \in S \ \text{ and } \ \text{dist}(y, \text{aco}(S)) < \varepsilon/8.$$

This implies that $x_\lambda \in S_{x_\lambda^*} = S(x_\lambda^*/\|x_\lambda^*\|, \varepsilon)$ and produces a finite number of elements $\{u^i\}_{i=1}^n \subset S$ with $u^i = (u_\delta^i)$ and a finite number of scalars $\{\lambda_i\}_{i=1}^n$ with $\sum_{i=1}^n |\lambda_i| = 1$ such that

$$\|y_\lambda - \sum_{i=1}^n \lambda_i u_\lambda^i\| + \sum_{\delta\neq\lambda} \|\sum_{i=1}^n \lambda_i u_\delta^i\| < \varepsilon/8. \tag{3.5}$$

Set

$$I = \{i \in \{1, \cdots, n\} : \|u_\lambda^i\| > 1/2 - \varepsilon/2\}.$$

We clearly have from (3.5) that $\|\sum_{i=1}^n \lambda_i u_\lambda^i\| > 1 - \varepsilon/8$. We then deduce from this that $\sum_{i\in I} |\lambda_i| \geq 1 - \varepsilon/4$. The same technique in Theorem 2.11 thus proves that

$\widetilde{u^i_\lambda} = u^i_\lambda/\|u^i_\lambda\| \in S_{x^*_\lambda}$ for all $i \in I$, and

$$\|y_\lambda - \sum_{i\in I} \widetilde{\lambda_i}\widetilde{u^i_\lambda}\| < \varepsilon \tag{3.6}$$

where $\widetilde{\lambda_i} = \lambda_i/(\sum_{i\in I}|\lambda_i|)$. For (3.6) we need the inequality

$$\|\sum_{i\in I}\lambda_i u^i_\lambda - \sum_{i\in I}\lambda_i\widetilde{u^i_\lambda}\| \le \sum_{i\in I}|\lambda_i|(1-\|u^i_\lambda\|) \le 1 - \|\sum_{i\in I}\lambda_i u^i_\lambda\| \le 3\varepsilon/8.$$

This finishes the proof. □

We next give an analogue of Proposition 3.12 for local-GL-spaces. The proof of this result is routine based on Theorem 2.11.

Proposition 3.13. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, and let $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_F$ where $F = c_0$ or l_1. Then E is a local-GL-space if and only if E_λ is a local-GL-space for every $\lambda \in \Lambda$.*

Proof. Let P_λ be the projection of E onto E_λ, and let I_λ be the injection of E_λ into E.

We first show the "if" part. Fix a separable subspace X of E. Then $P_\lambda(X) \subset E_\lambda$ is separable. Since E_λ is a local-GL-space, there is a GL-space $Y_\lambda \subset E_\lambda$ such that $P_\lambda(X) \subset Y_\lambda$. Then $Y = [\bigoplus_{\lambda\in\Lambda} Y_\lambda]_F$ containing X is a subspace of E. Moreover it follows from Theorem 2.11 that Y is a GL-space, and hence E is a local-GL-space.

Now let us deal with the "only if" part. Given $\lambda \in \Lambda$, let X_λ be a separable subspace of E_λ. Since E is a local-GL-space, there is a GL-space Y such that $I_\lambda(X_\lambda) \subset Y \subset E$. Note from Theorem 2.11 that $Y_\lambda = P_\lambda(Y)$ is a GL-space such that $X_\lambda \subset Y_\lambda \subset E_\lambda$. Thus E_λ is a local-GL-space. □

A similar analysis as the above proposition yields the following result.

Proposition 3.14. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of local-GL-spaces and let $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_{l_\infty}$. Then E is a local-GL-space.*

As immediate consequences of the propositions above, we obtain that:

Corollary 3.15. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of local-GL-spaces. Then the space $E = [\bigoplus_\lambda E_\lambda]_F$, where $F = c_0, l_1$ or l_∞ has the MUP.*

Throughout this paper, we can see that the geometry properties, isometric extension, and even the numerical index on unit spheres have harmonious inner relationship and may provide a possible way to solve the isometric extension problem in more general case. Note that there exist examples of Banach spaces with numerical index 1 but not lush spaces (see [11, Remark 4.2]). Then the first natural question to ask is the following:

Question 3.16. Does every Banach space with numerical index 1 have the MUP?

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